April 18, 2017

Tuesday, April 18, 2017 9:24 AM

Definition. Let S, X be topological spaces. Two continuous maps $f, g : S \longrightarrow X$ are **homotopic** if $\exists$ continuous $H : S \times [0,1] \longrightarrow X$, called a **homotopy** from f to g, such that

$$H(s,0) = f(s), \quad H(s,1) = g(s), \quad \forall\, s \in S$$

Notation. $f \simeq g$ or $f \overset{H}{\simeq} g$

Intuitively, for $t \in [0,1]$, let $h_t : S \longrightarrow X$ where $h_t(s) = H(s,t)$, $s \in S$

Then h_t is a continuous family of continuous maps from S to X such that $h_0 \equiv f$ and $h_1 \equiv g$.

It can be seen as a continuous way of changing f to g; or "deforming" the image $f(S)$ to $g(S)$ in the space X.

Exercise. Let $C(S,X)$ be the set of all continuous maps from S to X. Then $\simeq$ is an equivalence relation on $C(S,X)$.

Definition. $[S,X] = C(S,X)/\simeq$ is the quotient set containing **homotopy classes** of maps.

04-18-p2

Tuesday, April 18, 2017 9:24 AM

Example ① Let $f, g : S = [0, \pi] \longrightarrow X = \mathbb{R}^2$ be

$$f(s) = \sin(s) \quad \text{and} \quad g(s) = \cos(s)$$

Then $f \simeq g$ by the homotopy

$$H(s,t) = \sin\left(s + \frac{t\pi}{2}\right)$$

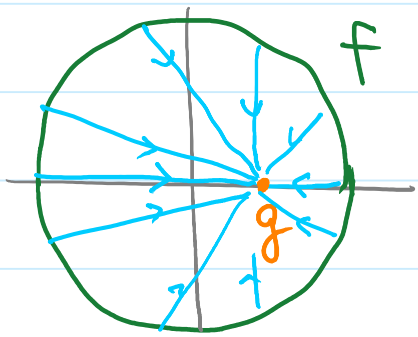
Example ② From the above, one would expect that any two maps f, g from $[a,b] \longrightarrow \mathbb{R}$ are homotopic.

Yes, e.g., $H(s,t) = (1-t)f(s) + t\,g(s)$

Example ③ Let $S = S^1 = \{w \in \mathbb{C} : |w| = 1\}$ and $f, g : S = S^1 \longrightarrow X = \mathbb{R}^2$ be

$$f(w) = w \quad ; \quad g(w) = \tfrac{1}{2} + 0i = \tfrac{1}{2}$$

$$S^1 \xrightarrow{f,\, g}$$



$$H(s,t) = (1-t)s + t \cdot \tfrac{1}{2}$$

Example ④ $S = S^1 = \{w \in \mathbb{C} : |w| = 1\}$
and now $X = \mathbb{C} \setminus \{0\}$; f, g are same as above.
In this case, the previous homotopy does not work because it goes over the origin.
It is expected that $f \not\simeq g$; but at this moment a proof is not ready.

Definition. Let $x_0 \in X$ and $c : S \to X$ be the constant map onto x_0, i.e., $c(s) = x_0$ $\forall s \in S$.
A continuous map $f : S \to X$ is null homotopic or homotopically trivial if $f \simeq c$ for some $x_0 \in X$.

Fact. Any map $S \to \mathbb{R}^n$, $n \geq 1$ is null homotopic.
In other words, $[S, \mathbb{R}^n] = C(S, \mathbb{R}^n)/\simeq = \{[c]\}$.

Definition. A subset $X \subset \mathbb{R}^n$ is star-shaped if $\exists\, x_0 \in X$ $\forall x \in X$ the straight line

$$\{(1-t)x + t x_0 : t \in [0,1]\} \subset X$$

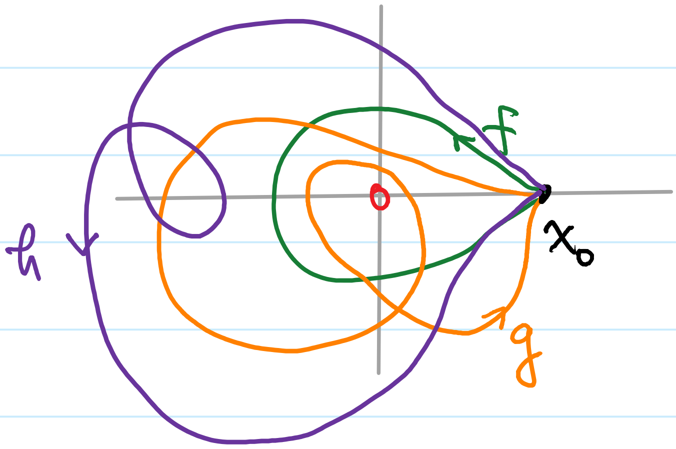
Obviously, a convex set is star-shaped.

Exercise. Show that any map $S \to X$ where $X \subset \mathbb{R}^n$ is star-shaped is null homotopic.

Remark. Now, one may observe that the center is not homotopy but the set $[S, X]$ for standard S.

Example ⑤ Conider the following maps $f, g, h : S^1 \longrightarrow \mathbb{C} \setminus \{0\}$

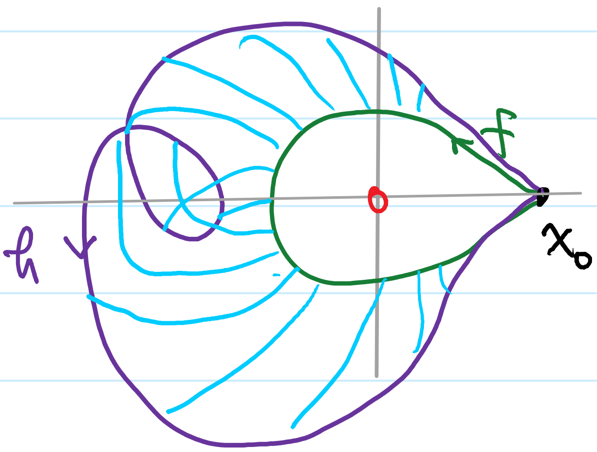
$S^1 \longrightarrow$



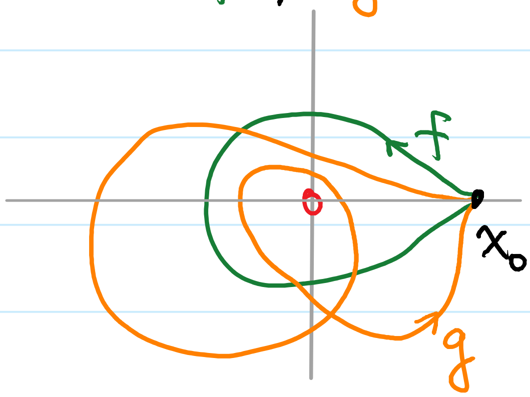
Intuitively, all of them are NOT null homotopic, also called homotopically non-trivial or homotopically essential

Pairwise comparing them, we see the following.

$f \simeq h$



$f \not\simeq g$



Definition In a topological space X with $x_0 \in X$. A **loop** in X at x_0 is a continuous path $\gamma : [0,1] \longrightarrow X$ with

$$\underbrace{\gamma(0) = \gamma(1) = x_0}_{\text{same starting and terminal point}}$$

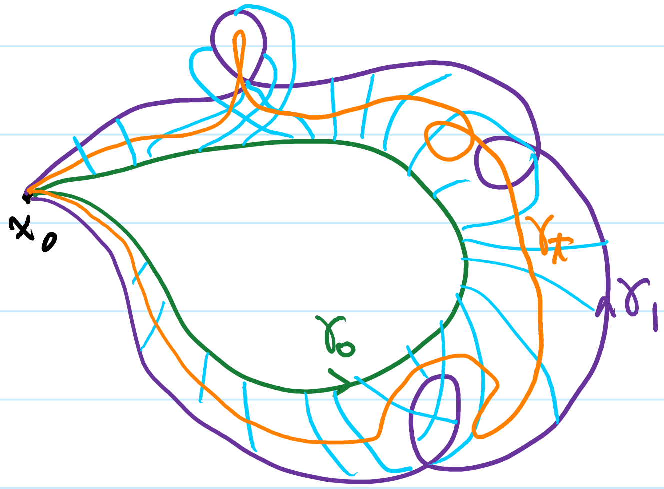
For loop parameter, use $[0,1]$ just for simplicity.

Definition. Two loops γ_0, γ_1 at x_0 are **loop homotopic** if $\exists$ continuous $L : [0,1] \times [0,1] \longrightarrow X$ such that

$L(s,0) = \gamma_0(s)$ $\forall\, s \in [0,1]$ (loop parameter)
$L(s,1) = \gamma_1(s)$

Just usual homotopy

and $\underbrace{L(0,t) = x_0 = L(1,t)}_{\text{Every } \gamma_t \text{ is a loop at } x_0}$ $\forall\, t \in [0,1]$ (time parameter)

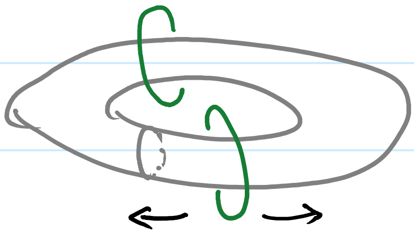


04-18-p6

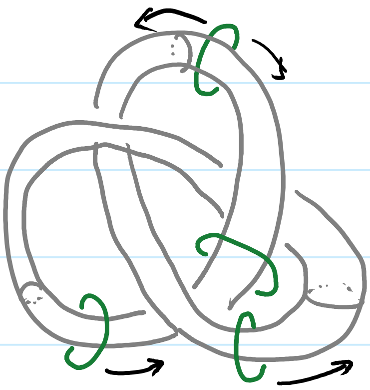
Wednesday, April 19, 2017 8:09 PM

Example The need to "nail" a point

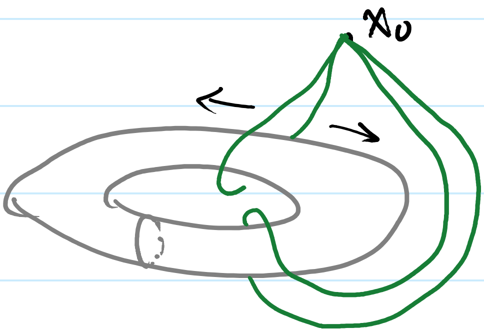
Complement of a circle

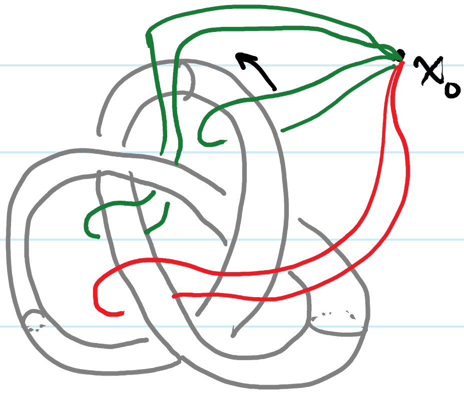


Complement of a knot



In both cases above, there is only one homotopy class. But in the cases below, more than one in the knot complement





Theorem

* Homotopy is an equivalence relation on $C(S, X) = \{\text{all continuous maps } S \to X\}$
* Loop homotopy is an equivalence relation on $\{\text{loops in } X \text{ at } x_0\}$

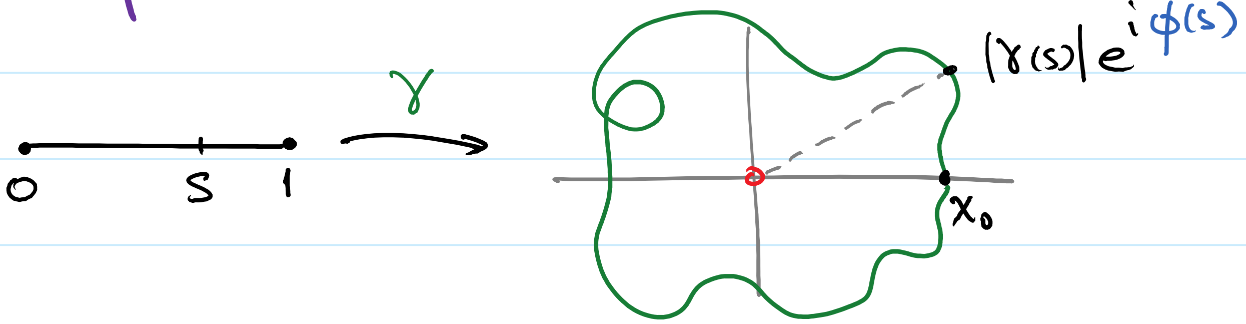
Definition. $\pi_1(X, x_0) = \{\text{loops at } x_0 \text{ in } X\} / \text{loop htpy}$

It is called **Fundamental group** as there is a group structure on it.

Example ① $X = \mathbb{R}^n$, $n \geq 1$, every map is homotopic to constant, $\therefore$ $\pi_1(\mathbb{R}^n) = \langle 1 \rangle$

Trivial group

Example ② $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0,0)\}$



We try to write $\gamma(s) = |\gamma(s)| e^{i\phi(s)}$, polar form.

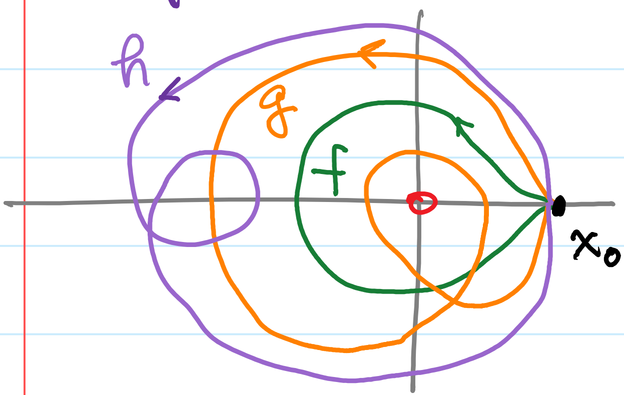
Problem. Whether $\phi(s)$ can be continuously defined for $s \in [0,1]$.

* For small $\varepsilon > 0$, $\phi(s)$ is good, $s \in [s_0 - \varepsilon, s_0 + \varepsilon]$
* $[0,1]$ and thus the image γ is compact,
  Thus, $[0,1]$ can be finitely covered by intervals where $\phi(s)$ is continuously defined.
* Starting from $\phi(0)$, inductively define $\phi(s)$.
* Since $\gamma(1) = \gamma(0)$, must have

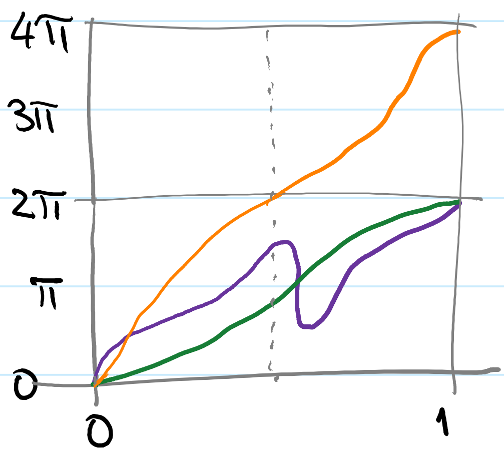
$$\phi(1) - \phi(0) = \text{a multiple of } 2\pi = 2w\pi, \quad w \in \mathbb{Z}$$

winding number

Example.



The corresponding pictures of ϕ_f, ϕ_g, ϕ_h.



$w(f) = 1$, $w(h) = 1$

$w(g) = 2$

Theorem $\pi_1(\mathbb{C}\setminus\{0\}) \underset{=\!=}{\xrightarrow{\text{bijection}}} \mathbb{Z}$

Idea of Proof.

From above, every loop in $\mathbb{C}\setminus\{0\}$ at x_0,

$$\gamma \longmapsto w(\gamma), \text{ winding number}$$

$*$ $[\gamma] \longmapsto w(\gamma)$ is well-defined

$$\underset{\pi_1(\mathbb{C}\setminus\{0\})}{\overset{[\gamma]}{\cap}} \qquad \underset{\mathbb{Z}}{\overset{w(\gamma)}{\cap}}$$

i.e., if $\gamma_0 \overset{\text{loop}}{\simeq} \gamma_1$ then $w(\gamma_0) = w(\gamma_1)$

The loop homotopy gives a continuous family $\gamma_t(s) = |\gamma_t(s)| e^{i\phi_t(s)}$ such that

$$\phi_t(1) - \phi_t(0) = 2\pi w(\gamma_t)$$

$t \in [0,1] \longmapsto w(\gamma_t) \in \mathbb{Z}$ is continuous

$$\therefore w(\gamma_0) = w(\gamma_1)$$

* $\pi_1(\mathbb{C}\setminus\{0\}) \xrightarrow{w(\gamma)} \mathbb{Z}$ is onto

For any $n \in \mathbb{Z}$, take

$$\gamma(s) = e^{2n\pi i s}, \quad s \in [0,1]$$

Then $w(\gamma) = n$

* $\pi_1(\mathbb{C}\setminus\{0\}) \xrightarrow{w(\gamma)} \mathbb{Z}$ is 1-1

i.e., to show if α, β are loops at x_0 with $w(\alpha) = w(\beta)$ then $\alpha \overset{\text{loop}}{\simeq} \beta$.

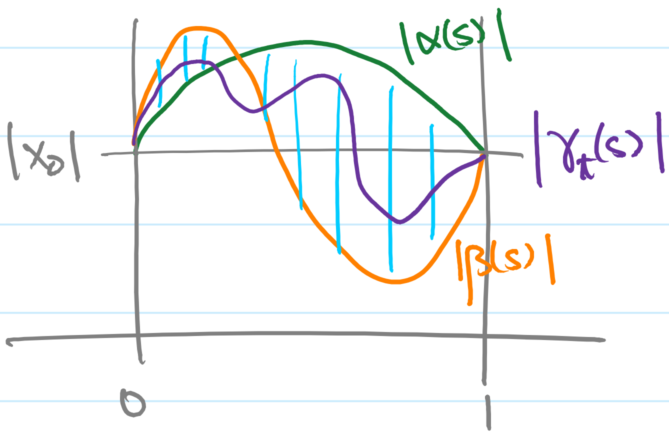
Write $\alpha(s) = |\alpha(s)| e^{i\phi_\alpha(s)}$ and

$$\beta(s) = |\beta(s)| e^{i\phi_\beta(s)}$$

As $\alpha(0) = \beta(0) = x_0$, choose $\phi_\alpha(0) = \phi_\beta(0)$

Since $|\alpha(0)| = |\beta(0)| = |x_0| = |\alpha(1)| = |\beta(1)|$ and $|\alpha(s)|, |\beta(s)| \in (0, \infty)$, it is easy to have a continuous family $|\gamma_t(s)|$ with $|\gamma_t(0)| = |x_0| = |\gamma_t(1)|$

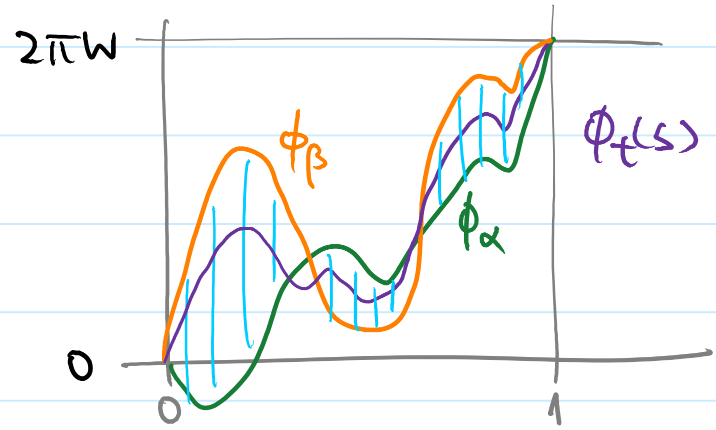
Illustration by



For $\phi_\alpha(s)$, $\phi_\beta(s)$, with choice of $\phi_\alpha(0)=\phi_\beta(0)$

Then $W(\alpha)=W(\beta) \Rightarrow \phi_\alpha(1)=\phi_\beta(1)$

The illustration becomes



One can have $\phi_t(s)$ such that

$\phi_\alpha(0)=\phi_t(0)=\phi_\beta(0)$, $\phi_\alpha(1)=\phi_t(1)=\phi_\beta(1)$ and

$$\phi_0 \equiv \phi_\alpha \quad , \quad \phi_1 \equiv \phi_\beta$$

Then we have a loop homotopy from α to β.

Theorem. $\pi_1(S^1) \underset{=}{\overset{\text{bijection}}{}} \mathbb{Z}$

Proof. For a loop γ in S^1, $|\gamma|=1$.

Thus, the above proof about homotopy between ϕ_α and ϕ_β is enough.